

ON DERIVED CATEGORIES OF NONMINIMAL ENRIQUES SURFACES

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ABSTRACT. By Orlov's formula, the derived category of blow up must contain the original variety as a semiorthogonal component. This arises an interesting question: does there exist a variety X such that $D^b(X)$ does not admit an exceptional collection of maximal length, but $D^b(\mathrm{Bl}_x X)$ admits such a collection? We give such an example where X is a minimal Enriques surface.

1. INTRODUCTION

This short note is to give an answer to the question posed in [7, Remark 3.12] on the existence of exceptional collections of maximal length in the derived categories of nonminimal Enriques surfaces. In his recent article [7], Vial proves that an algebraic surface S with $p_g = q = 0$ admits numerically exceptional collections of maximal length if and only if one of the following is true:

- (1) S is minimal and of Kodaira dimension $-\infty$ or 2;
- (2) S is one of the Dolgachev surfaces of type $X_9(2, 3)$, $X_9(2, 4)$, $X_9(3, 3)$, $X_9(2, 2, 2)$;
- (3) S is nonminimal.

By this criterion, an (minimal) Enriques surface never has an exceptional collection of maximal length, but there still remains a possibility that its blow up has an exceptional collection of maximal length. Indeed, it turns out that there exist such examples:

Theorem 1.1 (see Theorem 3.2). *There exist an Enriques surface S' such that the blowing up at a general point gives a surface S whose derived category admits a semiorthogonal decomposition of 13 line bundles together with a triangulated category \mathcal{A} satisfying $K_0(\mathcal{A}) = \mathbb{Z}/2\mathbb{Z}$.*

By the formula due to Orlov [6], we have two very different-looking semiorthogonal decompositions

$$D^b(S) = \langle \mathcal{O}_E(1), D^b(S') \rangle = \langle \mathcal{A}, E_1, \dots, E_{13} \rangle,$$

where E is the exceptional divisor of $S \rightarrow S'$. It seems a very intriguing question to ask how these semiorthogonal components can be compared.

In Section 2 we briefly explain notions related to the exceptional collections on algebraic surfaces. Section 3 deals with the construction method of the nonminimal Enriques surfaces which appear in Theorem 1.1 and the technical parts, including proofs, are discussed in Section 4. The theoretical backgrounds on Sections 3–4 are developed in [1], but these have been applied to simpler setup in this article. For this reason, we expect that this example is more comprehensible than the one in [1].

Notations 1.2.

- (1) Everything is defined over the field of complex numbers, except $Y_{\mathbb{Q}}$ in the proof of Lemma 4.9.
- (2) Let $\mu_r = \langle \zeta_r \rangle$ be the multiplicative group which is generated by the primitive r^{th} root of unity.

The group action

$$\mu_r \times \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad \zeta_r \cdot (x_1, \dots, x_n) = (\zeta_r^{a_1} x_1, \dots, \zeta_r^{a_n} x_n)$$

defines the quotient space \mathbb{C}^n/μ_n , which we will denote by $\mathbb{C}^n/\frac{1}{r}(a_1, \dots, a_n)$.

- (3) For a scheme T of finite type over \mathbb{C} and a point $P \in T$, $(P \in T)$ denotes the analytic germ.
- (4) Except stated otherwise, the equality between divisors indicates the linear equivalence relation. Also, we say D is effective if D is linearly equivalent to an effective divisor, or equivalently, $h^0(D) > 0$.

2. PRELIMINARIES

Let X be a nonsingular projective variety over \mathbb{C} and let $D^b(X)$ be the bounded derived category of coherent sheaves on X . An *exceptional collection* is an ordered collection of objects $E_1, \dots, E_k \in D^b(X)$ satisfying the following conditions:

$$\mathrm{Hom}_{D^b(X)}(E_i, E_j[p]) \begin{cases} \mathbb{C} & i = j, p = 0 \\ 0 & i = j, p \neq 0 \\ 0 & i > j \end{cases}$$

This notion is motivated from the decomposition problem in derived categories. In general, a triangulated category \mathcal{T} admits a *semiorthogonal decomposition* $\langle \mathcal{T}_1, \dots, \mathcal{T}_k \rangle$ if

- (1) $\mathcal{T}_1, \dots, \mathcal{T}_k$ are full triangulated subcategories of \mathcal{T} ;
- (2) the smallest full triangulated subcategory containing $\mathcal{T}_1, \dots, \mathcal{T}_k$ is \mathcal{T} ;
- (3) $\mathrm{Hom}_{\mathcal{T}}(T_i, T_j) = 0$ for each $i > j$ and $T_i \in \mathcal{T}_i, T_j \in \mathcal{T}_j$.

If $\mathcal{T} = D^b(X)$ and E_1, \dots, E_k is an exceptional collection, then there exists a semiorthogonal decomposition

$$\mathcal{T} = \langle \mathcal{A}, E_1, \dots, E_k \rangle$$

where $\mathcal{A} = \langle E_1, \dots, E_k \rangle^\perp$ is the full triangulated subcategory generated by the objects

$$\{A \in D^b(X) : \mathrm{Hom}_{D^b(X)}(E_i, A[p]) = 0, i = 1, \dots, k, p \in \mathbb{Z}\}$$

In practical situations, the Hom-groups in the derived category has a geometric interpretation. Indeed, for any coherent sheaf \mathcal{F} on X , we can regard \mathcal{F} as an objects in $D^b(X)$ by considering the complex $(\dots \rightarrow 0 \rightarrow \mathcal{F} \rightarrow 0 \rightarrow \dots)$ concentrated at degree zero, then for coherent sheaves \mathcal{F}_1 and \mathcal{F}_2 one has

$$\mathrm{Hom}_{D^b(X)}(\mathcal{F}_1, \mathcal{F}_2[p]) \simeq \mathrm{Ext}_X^p(\mathcal{F}_1, \mathcal{F}_2). \quad (2.1)$$

An exceptional object in $D^b(X)$ contributes a \mathbb{Z} -direct summand in the group $K_0(X)$. Thus, if E_1, \dots, E_k is an exceptional collection, then $K_0(X) = \mathbb{Z}^{\oplus k} \oplus K_0(\mathcal{A})$ where $\mathcal{A} = \langle E_1, \dots, E_k \rangle^\perp$. For this reason, the length of an exceptional collection is bounded by the rank of $K_0(X)$. If X is an algebraic surface with $\mathrm{CH}^2(X) \simeq \mathbb{Z}$, then it is known that $K_0(X) \simeq \mathbb{Z}^{\oplus 2} \oplus \mathrm{Pic} X$ (see [3, Lemma 2.7]). Hence, for S as in Theorem 1.1,

$$K_0(S) \simeq \mathbb{Z}^{\oplus 2} \oplus \mathrm{Pic} S \simeq \mathbb{Z}^{\oplus 13} \oplus \mathbb{Z}/2\mathbb{Z},$$

thus the length of any exceptional collection does not exceed 13. Also, once we establish such a collection, say E_1, \dots, E_{13} , then the orthogonal category $\mathcal{A} := \langle E_1, \dots, E_{13} \rangle^\perp$ must satisfy $K_0(\mathcal{A}) = \mathbb{Z}/2\mathbb{Z}$, and in particular, $\mathcal{A} \not\simeq 0$.

3. CONSTRUCTION METHOD

We begin with explaining the method to construct Enriques surfaces by \mathbb{Q} -Gorenstein smoothing. The method is originally developed in [5]. Also, the paper contains the construction of Enriques surfaces as an example (see [5, Example 2]). Here, we give a minimal description to establish the notations for the future use. Let $h_1, h_2 \in S := \mathbb{C}[x, y, z]$ be homogeneous cubics which define nodal cubics on the plane. Assume $h_1 \cap h_2$ are nine distinct points. Then the pencil $\mathfrak{p} := |\lambda_1 h_1 + \lambda_2 h_2|$ defines $\varphi_{\mathfrak{p}}: \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ whose resolution of indeterminacy is the rational elliptic fibration $f': Y' \rightarrow \mathbb{P}^1$. There are two special fibers which corresponds to $(h_i = 0)$ for $i = 1, 2$. Let $Y \rightarrow Y'$ be the blow up at the nodal points of $(h_i = 0)$. It ends up with the rational elliptic surface $f: Y \rightarrow \mathbb{P}^1$ with the following dual graph of divisors.

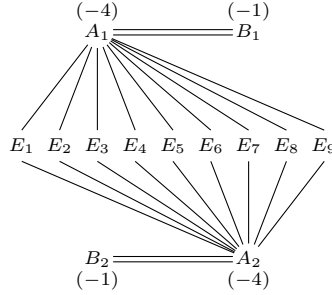


Figure 3.1. Divisors on Y and their intersections

Here, A_i is the proper transform of $(h_i = 0)$ along $Y \rightarrow \mathbb{P}^2$, and B_i is the exceptional divisor obtained by the blowing up $Y \rightarrow Y'$. Also, E_1, \dots, E_9 are the divisors which appear in the blowing up $Y' \rightarrow \mathbb{P}^2$. An edge between two nodes implies that corresponding divisors meet with intersection number 1. For example, it can be read $(A_i \cdot B_i) = 2$ from the graph. From Y , we can produce Enriques surfaces as follows.

Step 1. Contract A_1 and A_2 to gain a singular surface X with two $\frac{1}{4}(1, 1)$ singularities.

Step 2. Consider a \mathbb{Q} -Gorenstein smoothing $\mathcal{X}/(0 \in \Delta)$ of X , *i.e.* a proper flat morphism $\mathcal{X} \rightarrow (0 \in \Delta)$ such that $\mathcal{X}_0 \simeq X$ and \mathcal{X}_t is smooth for general $t \in \Delta \setminus \{0\}$.

Step 3. For general $t \in \Delta \setminus \{0\}$, $S := \mathcal{X}_t$ is an Enriques surface.

One possible way to perform a blow up S is to blow up the surface Y in advance, and proceeds to Steps 1–3 described above. Hence, we add:

Step 0. Blow up a point in $Y \setminus (A_1 \cup A_2 \cup B_1 \cup B_2 \cup E_1 \cup \dots \cup E_9)$.

By a slight abuse of notations, we keep call the resulting surface Y and the respective divisors $A_1, A_2, B_1, B_2, E_1, \dots, E_9$. Also, let E_0 be the exceptional divisor produced in Step 0.

Notations 3.1. Let $\pi: Y \rightarrow X$ be the contraction of A_1, A_2 , let $p: Y \rightarrow \mathbb{P}^2$ be the blow down morphism, and let $H \subset \mathbb{P}^2$ be a line. Then, $\text{Pic } Y$ is the free abelian group generated by

$$\{p^*H, E_0, E_1, E_2, \dots, E_9, B_1, B_2\}.$$

Also define divisors $Q := p^*(2H)$, $\ell_i := p^*H - E_i$ for $i = 1, \dots, 9$, and

$$D_i = \begin{cases} 0 & i = 0 \\ -\ell_i + E_0 + B_1 & 1 \leq i \leq 9 \\ -B_1 + E_0 & i = 10 \\ -Q + 3E_0 + 2B_1 & i = 11 \\ 2D_{11} & i = 12. \end{cases}$$

The push forward along π defines the \mathbb{Q} -Cartier Weil divisors on X .

Our main claim in this article is the following.

Theorem 3.2. *Assume h_1, h_2 are general. There exist divisors $D_i^{\mathfrak{g}} \in \text{Pic } S$ ($i = 0, \dots, 12$), which correspond to D_i , such that*

$$\mathcal{O}_S(D_0^{\mathfrak{g}}), \mathcal{O}_S(D_1^{\mathfrak{g}}), \dots, \mathcal{O}_S(D_{12}^{\mathfrak{g}})$$

is an exceptional collection in $D^b(S)$.

4. THE PROOF

Our aim is to find divisors on S which are comparable to the divisors on X . One of the natural attempts is to find a line bundle $\mathcal{L} \in \text{Pic } \mathcal{X}$ so that the line bundles $\mathcal{L}|_S$ and $\mathcal{L}|_X$ share some information through \mathcal{L} . Unfortunately, this cannot be done for some line bundles on S .

Proposition 4.1. *There exists a short exact sequence*

$$0 \rightarrow \text{Pic } S \rightarrow \text{Cl } X \rightarrow H_1(M_1, \mathbb{Z}) \oplus H_1(M_2, \mathbb{Z}),$$

where M_i is the Milnor fiber of the smoothing $(P_i \in \mathcal{X})/(0 \in \Delta)$. In particular, the image of $\text{Pic } S \rightarrow \text{Cl } X$ is exactly the set of Weil divisors D_X on X such that

$$(D.A_i) \equiv 0 \pmod{2} \quad i = 1, 2$$

where $D \in \text{Pic } Y$ is a proper transform of D_X along π .

Proof. For the proof, we refer to the arguments in [1, §3.1]. □

Indeed, we have divisors on Y which satisfies $(D.A_i) \equiv 0 \pmod{2}$ but π_*D is not Cartier, thus it is impossible to find $\mathcal{L} \in \text{Pic } \mathcal{X}$ such that $\mathcal{L}|_X = \mathcal{O}(\pi_*D)$. We need a workaround, which modifies the total space \mathcal{X} so that it is able to assign a line bundle on modified family $\tilde{\mathcal{X}}$ to each line bundle on S . This birational modification trick is developed in [4, §3]. There are two singularities, say P_1, P_2 in X , and these are isomorphic to $(0 \in \mathbb{C}_{u,v}^2/\frac{1}{4}(1,1))$. By the change of variables $x = u^2, y = v^2, z = uv$, we get

$$(P_i \in X) \simeq (0 \in (xy = z^2)) \subset (0 \in \mathbb{C}_{x,y,z}^3/\frac{1}{2}(1,1,1)).$$

The versal \mathbb{Q} -Gorenstein deformation is given by

$$\mathcal{X}^{\text{ver}} := (xy = z^2 + t) \subset \mathbb{C}_{x,y,z}^3/\frac{1}{2}(1,1,1) \times \Delta_t^{\text{ver}},$$

where $\Delta_t^{\text{ver}} \subset \mathbb{C}$ is a small complex disk centered at the origin. Locally, the ambient space $\mathbb{C}_{x,y,z}^3/\frac{1}{2}(1,1,1) \times \Delta_t^{\text{ver}}$ can be identified to the toric variety associated with the fan Σ generated by the standard basis of \mathbb{Z}^4 inside the lattice $N = \mathbb{Z}^4 + \mathbb{Z} \cdot \frac{1}{2}(1,1,1,2)$. Let $\tilde{\Sigma}$ be the fan obtained by adding the ray $\mathbb{Z} \cdot \frac{1}{2}(1,1,1,2)$ to Σ . The resulting toric variety $\tilde{\mathbb{C}}$ admits the birational morphism $\Phi': \tilde{\mathbb{C}} \rightarrow \mathbb{C}_{x,y,z}^3/\frac{1}{2}(1,1,1) \times \Delta_t^{\text{ver}}$.

Let $\tilde{\mathcal{X}}^{\text{ver}}$ be the proper transform of \mathcal{X}^{ver} and let $\Phi^{\text{ver}}: \tilde{\mathcal{X}}^{\text{ver}} \rightarrow \mathcal{X}^{\text{ver}}$ be the birational morphism induced by Φ' .

Proposition 4.2 (cf. [4, §3]). *Let $\Phi: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$ be the pullback of Φ^{ver} along $\mathcal{X}/(0 \in \Delta) \rightarrow \mathcal{X}^{\text{ver}}/(0 \in \Delta^{\text{ver}})$. Then, $\tilde{\mathcal{X}}$ satisfies the following properties;*

- (1) *Over P_i , $W_i := \Phi^{-1}(P_i)$ is a projective plane and Φ is an isomorphism outside $\{P_1, P_2\}$.*
- (2) *The central fiber over $0 \in \Delta$ is the union $Y \cup W_1 \cup W_2$, where Y is the rational elliptic surface introduced above, and the scheme-theoretic intersection $W_i \cap Y$ is realized as A_i in Y , while it is a conic in W_i .*

Now, suppose that a divisor $D \in \text{Pic } Y$ satisfies the conditions in Proposition 4.1. Let $2d_i := (D.A_i)$, then since A_i is a conic curve in W_i , $\mathcal{O}_Y(D)|_{A_i} \simeq \mathcal{O}_{W_i}(d_i)|_{A_i}$. Let \mathcal{D}_0 be the glueing of $\mathcal{O}_Y(D)$, $\mathcal{O}_{W_1}(d_1)$, $\mathcal{O}_{W_2}(d_2)$, i.e. the kernel of

$$\mathcal{O}_Y(D) \oplus \mathcal{O}_{W_1}(d_1) \oplus \mathcal{O}_{W_2}(d_2) \rightarrow \mathcal{O}_{A_1}(2d_1) \oplus \mathcal{O}_{A_2}(2d_2), \quad (s, s_1, s_2) \mapsto (s - s_1, s - s_2).$$

Then, it can be easily proved that \mathcal{D}_0 is an exceptional line bundle on the reducible surface $\tilde{\mathcal{X}}_0 = Y \cup W_1 \cup W_2$. It is well-known that an exceptional vector bundle extends to a small neighborhood of deformation, thus shrinking Δ , we can say that there exists a line bundle $\tilde{\mathcal{D}}$ such that $\tilde{\mathcal{D}}|_{\tilde{\mathcal{X}}_0} = \mathcal{D}_0$. Now, $\tilde{\mathcal{D}}|_S$ is a line bundle on S .

Notations 4.3. For a divisor $D \in \text{Pic } Y$ as above, the divisor associated with the line bundle $\tilde{\mathcal{D}}|_S$ is denoted by $D^{\mathfrak{g}}$.

Since $\tilde{\mathcal{D}}$ is a flat family of line bundles, we have $\chi(D^{\mathfrak{g}}) = \chi(\tilde{\mathcal{D}}_0)$. The latter can be computed via the short exact sequence

$$0 \rightarrow \tilde{\mathcal{D}}_0 \rightarrow \mathcal{O}_Y(D) \oplus \mathcal{O}_{W_1}(d_1) \oplus \mathcal{O}_{W_2}(d_2) \rightarrow \mathcal{O}_{A_1}(2d_1) \oplus \mathcal{O}_{A_2}(2d_2) \rightarrow 0. \quad (4.2)$$

By Riemann-Roch formula,

$$\chi(\tilde{\mathcal{D}}_0) = \chi(D) + \frac{1}{2}d_1(d_1 - 1) + \frac{1}{2}d_2(d_2 - 1). \quad (4.3)$$

Lemma 4.4. *For any divisor $D \in \text{Pic } Y$ such that there exists an associated divisor $D^{\mathfrak{g}} \in \text{Pic } S$,*

$$(D^{\mathfrak{g}}.K_S) = (D.E_0).$$

Proof. Since K_S is numerically equivalent to $E_0^{\mathfrak{g}}$, it suffices to prove that $(D^{\mathfrak{g}}.E_0^{\mathfrak{g}}) = (D.E_0)$. By Riemann-Roch formula, one can compute

$$(D^{\mathfrak{g}}.E_0^{\mathfrak{g}}) = \chi(D^{\mathfrak{g}} + E_0^{\mathfrak{g}}) - \chi(D^{\mathfrak{g}}).$$

Let $\tilde{\mathcal{D}}_0$ be the line bundle obtained by glueing $\mathcal{O}_Y(D)$, $\mathcal{O}_{W_1}(d_1)$, $\mathcal{O}_{W_2}(d_2)$, and $\tilde{\mathcal{E}}_0$ be the line bundle obtained by glueing $\mathcal{O}_Y(E_0)$, \mathcal{O}_{W_1} , \mathcal{O}_{W_2} . Then, using the formula (4.3) we get

$$\begin{aligned} (D^{\mathfrak{g}}.E_0^{\mathfrak{g}}) &= \chi(D^{\mathfrak{g}} + E_0^{\mathfrak{g}}) - \chi(D^{\mathfrak{g}}) = \chi(\tilde{\mathcal{D}}_0 \otimes \tilde{\mathcal{E}}_0) - \chi(\tilde{\mathcal{D}}_0) \\ &= \left(\chi(D + E_0) + \frac{1}{2}d_1(d_1 - 1) + \frac{1}{2}d_2(d_2 - 1) \right) - \left(\chi(D) + \frac{1}{2}d_1(d_1 - 1) + \frac{1}{2}d_2(d_2 - 1) \right) \\ &= \chi(D + E_0) - \chi(D) = (D.E_0). \end{aligned} \quad \square$$

Combining all together, we get the intersection formula:

$$(D^g)^2 = (D.E_0) + 2\chi(D) + d_1(d_1 - 1) + d_2(d_2 - 1) - 2.$$

Note that all the information from the right hand side can be read in Y . Now, it is just a matter of computations to derived the following intersection table: let $1 \leq i \neq j \leq 9$

	Q^g	ℓ_i^g	ℓ_j^g	B_1^g	E_0^g
Q^g	22	10	10	3	0
ℓ_i^g	10	2	3	1	0
ℓ_j^g	10	3	2	1	0
B_1^g	3	1	1	0	0
E_0^g	0	0	0	0	-1

(4.4)

Note that $B_1^g - B_2^g$ is numerically trivial, so the above table also includes the intersections involving B_2^g .

Proposition 4.5. *The intersection matrix of divisors $\{D_i^g\}_{i=1,\dots,11}$ is given by*

$$((D_i^g.D_j^g))_{1 \leq i,j \leq 11} = \begin{bmatrix} -1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -1 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Also, it holds that $K_S =_{\text{num}} D_1^g + \dots + D_{10}^g - 3D_{11}^g$, where $=_{\text{num}}$ indicates the numerical equivalence relation.

Proof. The intersection table is obtained immediately from (4.4). Furthermore,

$$\begin{aligned} \sum_{i=1}^{10} D_i^g - 3D_{11}^g &= (-\sum_{i=1}^9 \ell_i^g + 8B_1^g + 10E_0^g) - 3(-Q^g + 2B_1^g + 3E_0^g) \\ &= 3Q^g - \sum_{i=1}^9 \ell_i^g + 2B_1^g + E_0^g, \end{aligned}$$

and $\sum_{i=1}^9 \ell_i = (p^*(6H))^g + (p^*(3H) - \sum_{i=1}^9 E_i))^g = 3Q^g + A_0^g$ where A_0^g is the general elliptic fiber of S induced by the elliptic fibration of $Y \rightarrow \mathbb{P}^1$. This leads to

$$\sum_{i=1}^{10} D_i^g - 3D_{11}^g = -A_0^g + 2B_1^g + E_0^g.$$

Since $A_0^g = 2B_1^g$, we have $\sum_{i=1}^{10} D_i^g - 3D_{11}^g = E_0^g =_{\text{num}} K_S$. □

Corollary 4.6. *For $0 \leq j < i \leq 12$, $\chi(-D_i^g + D_j^g) = 0$.*

Proof. This is an immediate consequence of Proposition 4.5 and Riemann-Roch theorem. □

To prove Theorem 3.2, we have to prove that $h^p(-D_i^g + D_j^g) = 0$ for each p and $0 \leq j < i \leq 12$. By the above corollary, it suffices to prove only for $p = 0, 2$. By Serre duality, $h^2(-D_i^g + D_j^g) = h^0(K_S + D_i^g - D_j^g)$, thus understanding h^0 of divisors is enough to prove Theorem 3.2. This can be done using the short exact sequence (4.2); it is easy to see that $h^0(\mathcal{O}_{W_i}(d_i)) \rightarrow h^0(\mathcal{O}_{A_i}(2d_i))$ is always surjective, thus

$$h^0(D^g) \leq h^0(\tilde{D}_0) = h^0(D) + h^0(\mathcal{O}_{W_1}(d_1)) + h^0(\mathcal{O}_{W_2}(d_2)) - h^0(\mathcal{O}_{A_1}(2d_1)) - h^0(\mathcal{O}_{A_2}(2d_2)). \quad (4.5)$$

Before proceed to the proof, we give one remark explaining why we need to use a computer-based approach.

Example 4.7. The divisor $-D_9^{\mathfrak{g}} + D_0^{\mathfrak{g}} = (p^*H - E_9 - E_0 - B_1)^{\mathfrak{g}}$ can be obtained by deforming $D := p^*H - E_9 - E_0 - B_1$. Since $(D.A_1) = 0$ and $(D.A_2) = 2$,

$$\begin{aligned} h^0(D^{\mathfrak{g}}) &\leq h^0(D) + h^0(\mathcal{O}_{W_1}) + h^0(\mathcal{O}_{W_2}(1)) - h^0(\mathcal{O}_{A_1}) - h^0(\mathcal{O}_{A_2}(2)) \\ &= h^0(D). \end{aligned}$$

Because of its divisor form, $h^0(D)$ depends on the configuration of $(h_i = 0)$. Indeed, $h^0(D) = 1$ if the points $p(E_9)$, $p(E_0)$ and $p(B_1)$ are colinear and is zero otherwise. If these three points are colinear, we just have $h^0(D^{\mathfrak{g}}) \leq 1$, thus we cannot see the desired vanishing. In this simple example, we can present the possible values of $h^0(D)$ together with exact criterion, but it is getting complicated for other pairs. For example, to conclude $h^0(-D_{12}^{\mathfrak{g}} + D_{10}^{\mathfrak{g}}) = 0$, we will see that it suffices to prove that $h^0(D) = 0$ where

$$p^*(16H) - 4 \sum_{i \leq 9} E_i - 6E_0 - 6A_2 - 6B_2.$$

This means that we have to show there is no plane curve of degree 16 which passes through $p(E_1), \dots, p(E_9)$ 4 times for each, $p(E_0)$ 6 times, $p(A_2)$ 6 times, and $p(B_2)$ 6 times.

Lemma 4.8. *Let D be a divisor of the form*

$$p^*(dH) - (\text{positive sum of } E_0, E_1, \dots, E_9, B_1, B_2). \quad (4.6)$$

For given particular h_1, h_2 , assume $h^0(D) = N$. Then, $h^0(D) \leq N$ for general h_1, h_2 .

Proof. As explained in Example 4.7, counting $h^0(D)$ reduces down to count the dimension of the space of plane curves passing through the prescribed positions. Consider the homogeneous equation $\sum_{\alpha} c_{\alpha} \mathbf{x}^{\alpha}$ of degree d , where $\alpha = (\alpha_x, \alpha_y, \alpha_z)$ is a 3-tuple with $\alpha_x + \alpha_y + \alpha_z = d$ and $\mathbf{x}^{\alpha} = x^{\alpha_x} y^{\alpha_y} z^{\alpha_z}$. Then the positional conditions given by D will impose linear conditions on (c_{α}) , thus we get a system of linear equations, say $M\mathbf{c} = 0$. Now, $h^0(D)$ is the dimension of the space of solutions of this system. If we perturb h_1, h_2 then it perturbs M , but the rank of matrices is a lower-semicontinuous function, so $\dim \ker M = h^0(D)$ is upper-semicontinuous with respect to h_1, h_2 . \square

Now it suffices to prove Theorem 3.2 for the particular choice of h_1, h_2 . However, since we use computer-based approach, we still need another obstruction to overcome. Imagine the situation that we are given a divisor D of the form (4.6), and h_1, h_2 are explicitly given. We have to tell the computer the conditions imposed by the plane curve p_*D as an ideal sheaf of \mathbb{P}^2 . The problem is that it is extremely hard (perhaps impossible) to find an example of h_1, h_2 such that the points corresponding to $E_1, \dots, E_9, B_1, B_2$ are defined over a subfield of \mathbb{C} which is solvable by radicals over \mathbb{Q} . Unless this is possible, we cannot define the explicit ideals in computers. This problem can be resolved by observing some symmetric nature between E_1, \dots, E_9 . In the end, we will see that it suffices to find cubics such that only $p(E_9), p(B_1), p(B_2)$ are defined over \mathbb{Q} .

Lemma 4.9. *Assume h_1, h_2 define general nodal plane cubics. Let D be a divisor in the rational elliptic surface Y , and assume that in the expression of D in terms of the \mathbb{Z} -basis $\{p^*H, E_1, \dots, E_9, A_2, B_2, B_3\}$, the coefficients of E_1, \dots, E_9 are same. Then, $h^p(D + E_i) = h^p(D + E_j)$ for any p and any $1 \leq i, j \leq 9$.*

Proof. The statement is a slight variation of [1, Lemma 5.8], so we do not give a precise proof here. The main idea is the following: first, pick h_1, h_2 so that

- (1) the point $p(E_0)$, the cubics $h_1, h_2 \in \mathbb{Q}[x, y, z]$, and the nodes of them are defined over \mathbb{Q} ;

- (2) the ideal (h_1, h_2) is prime in $\mathbb{Q}[x, y, z]$;
- (3) the points in the intersection is written as the form $[x_i, y_i, 1] \in \mathbb{P}_{\mathbb{C}}^2$, where x_1, \dots, x_9 are Galois conjugate to each other, y_1, \dots, y_9 are Galois conjugate to each other, and the irreducible polynomials of x_i and y_j are not the same.

Now, we take $\tau \in \text{Aut}(\mathbb{C}/\mathbb{Q})$ such that $\tau(x_i) = x_j$. By condition (1), Y is indeed defined over \mathbb{Q} , i.e. there exists a variety $Y_{\mathbb{Q}}$ defined over \mathbb{Q} such that $Y = Y_{\mathbb{Q}} \times_{\mathbb{Q}} \mathbb{C}$. Consider the automorphism $\tau_Y := \text{id}_{Y_{\mathbb{Q}}} \times \tau$ of Y . Then, τ_Y permutes E_1, \dots, E_9 (sending E_i to E_j), and fixes E_0, B_1, B_2 . Thus τ_Y fixes D . In particular,

$$h^p(D + E_j) = h^p(\tau_Y^*(D + E_j)) = h^p(D + E_i). \quad \square$$

By Lemmas 4.8 and 4.9, the main part of the proof reduces to the following statement:

Proposition 4.10. *There exists $h_1, h_2 \in \mathbb{C}[x, y, z]$ such that*

$$\mathcal{O}_S(D_0^{\mathfrak{g}}), \mathcal{O}_S(D_9^{\mathfrak{g}}), \mathcal{O}_S(D_{10}^{\mathfrak{g}}), \mathcal{O}_S(D_{11}^{\mathfrak{g}}), \mathcal{O}_S(D_{12}^{\mathfrak{g}})$$

is an exceptional collection in $D^b(S)$.

Proof. We choose $h_1 = (y - z)^2 z - x^3 - x^2 z$ and $h_2 = x^3 - 2xy^2 + 2xyz + y^2 z$, and pick $p(E_0) = [4, 9, 6]$. We denote the number $h^p(-D_i^{\mathfrak{g}} + D_j^{\mathfrak{g}})$ by $h_{i,j}^p$. First of all, we observe that the divisor $B_1^{\mathfrak{g}}$ is nef, hence any divisor $D^{\mathfrak{g}} \in \text{Pic } S$ with $(D^{\mathfrak{g}} \cdot B_1^{\mathfrak{g}}) < 0$ must have vanishing h^0 . Also, by Serre duality, $(D_i^{\mathfrak{g}} - D_j^{\mathfrak{g}} \cdot B_1^{\mathfrak{g}}) > 0$ implies $h_{i,j}^2 = 0$. Using this criterion, we get the vanishing of the following numbers:

$$h_{9,0}^2, h_{10,9}^0, h_{11,0}^2, h_{11,9}^2, h_{11,10}^2, h_{12,0}^2, h_{12,9}^2, h_{12,10}^2, h_{12,11}^2.$$

In what follows, we prove $h_{i,j}^p = 0$ for $i, j \in \{0, 9, 10, 11, 12\}$ with $j < i$ by taking suitable divisor $D \in \text{Pic } Y$ such that D deforms to either $-D_i^{\mathfrak{g}} + D_j^{\mathfrak{g}}$ of $K_S + D_i^{\mathfrak{g}} - D_j^{\mathfrak{g}}$, and by using (4.5).

result	choice of D
$h_{9,0}^0 = 0$	$p^*(H) - E_9 - E_0 - B_1$
$h_{10,9}^2 = 0$	$p^*(H) - E_9 + E_0 - B_1 - B_2$
$h_{11,0}^0 = 0$	$p^*(5H) - \sum_{i \leq 9} E_i - 3E_0 - 2B_1 - 2B_2$
$h_{11,9}^0 = 0$	$p^*(4H) - \sum_{i \leq 8} E_i - 2E_0 - B_1 - 2B_2$
$h_{11,10}^0 = 0$	$p^*(5H) - \sum_{i \leq 9} E_i - 2E_0 - 3B_1 - 2B_2$
$h_{12,0}^0 = 0$	$p^*(16H) - 4 \sum_{i \leq 9} E_i - 6E_0 - 6B_1 - 6B_2$
$h_{12,9}^0 = 0$	$p^*(12H) - 3 \sum_{i \leq 8} E_i - 2E_9 - 5E_0 - 6B_1 - 4B_2$
$h_{12,10}^0 = 0$	$p^*(10H) - 3 \sum_{i \leq 9} E_i - 5E_0 - 5B_1 - 6B_2$
$h_{12,11}^0 = 0$	same D as in $h_{11,0}^0$

For the first two in the table, it is easy to verify $h^0(D) = 0$ if the triples $(p(E_9), p(E_0), p(B_1))$ and $(p(E_9), p(B_1), p(B_2))$ are not colinear. For the rest part of the table, we use computer to find $h^0(D) = 0$. The Macaulay2 scripts can be found in [2]. \square

Proof of Theorem 3.2. The only thing remains to prove is $h_{i,j}^p = 0$ for $1 \leq j < i < 9$. This can be easily shown since (4.5) reads

$$h_{i,j}^0 \leq h^0(-E_i + E_j) = 0, \quad h_{i,j}^2 \leq h^0(K_Y + E_1 - E_2) = 0.$$

Lemmas 4.8, 4.9 and Proposition 4.10 imply that

$$\mathcal{O}_S(D_0^g), \mathcal{O}_S(D_1^g), \dots, \mathcal{O}_S(D_{12}^g)$$

is an exceptional collection after a slight perturbation of h_1 and h_2 . \square

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